

Small-Worlds, Mazes and Random Walks

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We establish a relationship between the Small-World behavior found in complex networks and a family of Random Walks trajectories using, as a linking bridge, a maze iconography. Simple methods to generate mazes using Random Walks are discussed along with related issues and it is explained how to interpret mazes as graphs and loops as shortcuts. Small-World behavior was found to be non-logarithmic but power-law in this model, we discuss the reason for this peculiar scaling.

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Small World (SW) graphs efficiently interpolate between regular and random graphs thanks to a small number pN of *shortcuts* (long-range connections) which are superimposed on a regular graph formed by N nodes. In random graphs, the mean minimal distance or diameter between all pairs of nodes in the system scales logarithmically with the system size, while it does it linearly in regular graphs. Much attention has been devoted recently to the topological properties of such graphs and to the effects that a SW-like connectivity may have on the properties of dynamical systems [1, 2, 3, 4].

There is a number of papers relating Random Walks (RW) and SW phenomenon [6, 7, 8, 9, 10, 11, 12, 13, 14]. In [8, 9] the authors examine RW on SW networks, in particular the probability of a random walker of being at the original site at a later time. Their interest stem from the motion of excitons over polymer chains, where steps between spatially close sites can connect regions far apart along the chemical backbone. However his model follows a standard SW network building [5]. Later the same authors [10], considering self-avoiding constraints, assume that the probability that two sites far apart along the backbone come close together in space is approximately an inverse power law of their mutual distance. In other studies RW have been employed as dynamical nodes to study dynamical SW effects [13, 14]. Furthermore and in another context, RW on the family of SW networks have been addressed where RW correspond to random spread of information over the network [11]. It was demonstrated that the average access time between

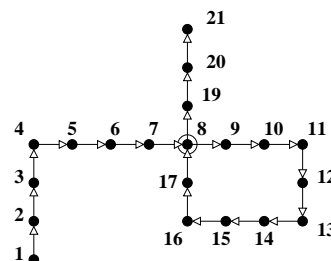


FIG. 1: An illustrative example of short-cut by loop in the RW path or maze: (a) The path **1-2-3-4-5-6-7-8-9-10-11-12-13-14-15-16-17-18-19-20-21** of $N = 21$ steps traced by a RW, can be interpreted as a maze of length $N = 21$. (b) To solve the maze is: to travel starting at **1**, and ending at **21**. One non-optimal solution is a travel of length 21. (c) At step 18 the path has a self-intersection with step 8, a loop **8-9-10-11-12-13-14-15-16-17-18**. (d) We can avoid this loop to solve optimally the maze. Then, the loop acts as a short-cut in the graph version of the maze: the node **8** is connected with the node **9** and **19**. (e) To solve efficiently the maze we use the minimal distance $L = 10$ between nodes **1** and **21**: **1-2-3-4-5-6-7-8-19-20-21**. (f) Then, the length of the maze is $N = 21$, but using the short-cut, it is solved in $L = 11$.

nodes for a SW geometry shows a crossover from regular to random behavior with increasing distance from the starting point of the RW. Average access time is important in any Markov process and is very relevant for the exploring and navigation of the WWW [12] that, as a scale-free network, it also shows the SW effect.

In this paper we study the properties of two dimensional complex mazes from the point of view of the SW theory. For this, it is important to explore new methods (such as the use of the properties of excitable media [17])

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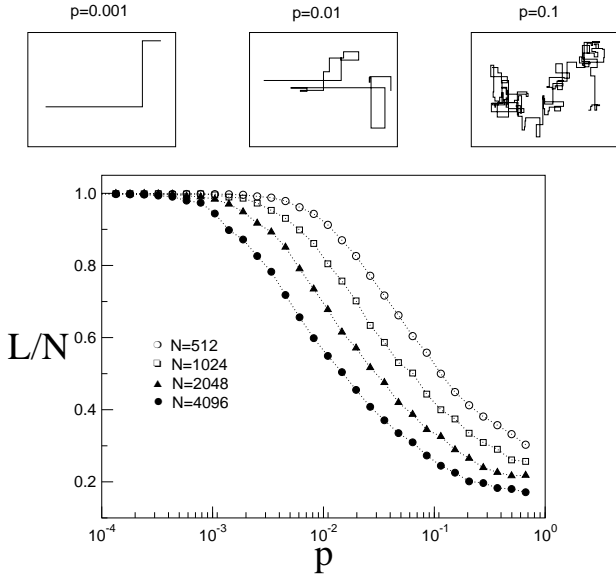


FIG. 2: Upper figures: Three examples of mazes generated by RWs with $p = 0.001$, $p = 0.01$, and $p = 0.1$ with a proper choice of the scale. Main figure: The effect of growth p on the minimal distance L normalized by N in the path graph for several sizes N and log-linear axes. For all simulations in this article, 31 points separated logarithmically between $p = 0.0001$ and $p = 2/3$ were considered. In all cases each point represents the mean of 1000 numerical experiments. The path sizes (or graph sizes) of the RW are: $N = 4096$ (filled circles), $N = 2048$ (triangles), $N = 1024$ (squares), and $N = 512$ (empty circles). We notice a size-effect in L similar to the one in the classic SW model that predicts a transition at $p \rightarrow 0$ and $N \rightarrow \infty$.

to find minimum-length paths in complex labyrinths. Navigational methods for solving mazes are widely applied in computer sciences for searching through data structures and the so-called depth-first search method is an example [16]. In this work we use non-reversing RW in two dimensions for generating mazes [18]. We may interpret the trajectory of the non-reversing RW in a plane as a directed graph in the following sense: each site reached at step i by the RW, is a node labeled by i . The nodes are connected in the step sequence, i. e.: $i \rightarrow i + 1$. But if a loop exists, for example i -step intersect with j -step then we connect $i \rightarrow j + 1$. Therefore the loops act as shortcuts in the graph. In figure 1 we show an example: a path traced by one of this non-reversing RW. The path is the maze and solving it consist in discovering the minimum number of steps, the minimum path from the first site to the last site traced by the RW. As we can see self-intersections are no avoided and so loops are permitted. Thus jumping the loops is a way for quickly reaching the exit.

To generate a specific maze we fix the number N of steps of the non-reversing RW and the probability $p \in [0, 2/3]$. In each step the RW will vary his direction with probability p : at right with probability $p/2$ or at left with the same probability. And with probability $1 - p$ the RW

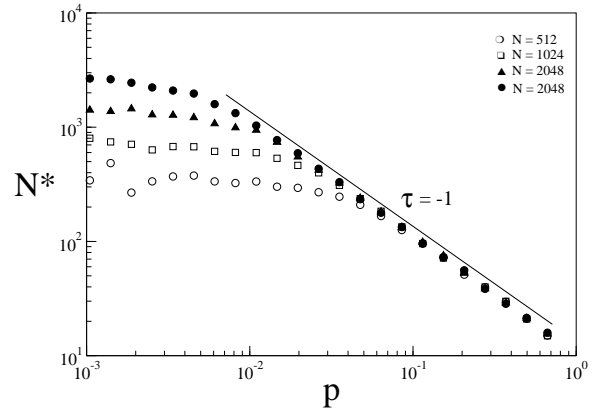


FIG. 3: Log-log graph showing the scaling of the number of RW steps N^* that takes to reach the first short-cut or the first loop as a function of p . The line with slope -1 is a guide for the eye. The linear fitting on the scaling region gives $\tau = -1.003 \pm 0.008$.

will not turn. In this manner we can construct a variety of mazes. From $p = 0$ that produces linear trajectories to $p = 2/3$ that gives intricate trajectories with equal probability to continue straight, turn right or turn left. Obviously the number of self-interactions grows with p . Each time a self-interaction occurs, there is a new loop.

In the upper insets of figure 2 we shown three cases, with a proper choice of the scale, for non-reversing RWs with $N = 1024$ steps. A very little value of p such as $p = 0.001$ produces a maze without loops. A value of $p = 0.01$ generates mazes with a moderate number of loops. Finally, a value of $p = 0.1$ generates very intricate mazes. Because increasing p imply an increase in the number of loops or equivalent short-cuts on a lineal trajectory or regular graph, we expect SW behavior in the model. We point out two differences with standard SW model: here the links are direct and the shortcuts are self-created by the system dynamics.

In the present work, the SW effect is used in the sense that the maze can be solved, walked from the start to the end, within few steps on the average, in spite of the relatively low number of loops present in the system. In the main plot at figure 2, the SW effect is evident. N represents the number of steps taken by the RW and so the maximum number of steps that takes going from the start to the end of the maze. If the number of path-intersections are regarded as shortcuts, then the minimal distance L is dramatically reduced with small increments of p .

Similarly to the SW standard model, it is possible to determine a persistence length N^* , that is the mean number of steps needed for producing the first loop or short-cut. In Figure 3, it is depicted the simple dependence regarding p : $N^* \sim p^{-1}$, that is equivalent to the average number of steps that a RW must take before turning for the first time.

On Figure 4, we observe that the following scaling re-

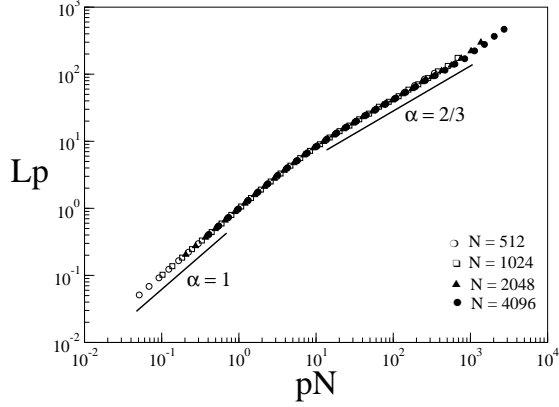


FIG. 4: Log-log graph showing the behavior of the scaling function F_1 . The lines with slope 1 and slope $2/3$ are a guide for the eye. The linear fitting on the scaling regions give $\alpha = 1.004 \pm 0.007$ and $\alpha = 0.668 \pm 0.003$ for $pN < 1$ and $pN > 1$ respectively.

lationship is satisfied:

$$L \sim p^{-1} F_1(pN). \quad (1)$$

If p^{-1} is proportional to the mean number of RW steps for producing one loop, $F_1(pN)$ can be interpreted as the mean number of loops in a system with size N and probability p in its building. The scaling function $F_1(x)$ behaves linearly for values of $x < 1$. That is, while pN -the average number of turnings- is less than one there are no shortcuts in the maze and the value of N matches L . When $x > 1$ the presence of shortcuts has an impact and the distance L is reduced following a power-law with an exponent value α equal to $2/3$ when the size of the maze is increase.

We depict in Figure 5 a collapse plot of the scaling function $F_2(x)$:

$$L \sim N F_2(pN). \quad (2)$$

$F_2(x)$ may be interpreted as the reduction factor in the start-end maze distance as a function of the size and the turning rate on the system given by p . Again, for values of $x < 1$, L and N take the same value. However, as N or p are increase in such a way that $pN > 1$ the ratio L/N decreases as a power law with an exponent value β equal to $-1/3$.

Both results confirm that:

$$L(N, p) \sim N, \quad (3)$$

for no loops or $pN < 1$. On the other hand:

$$L(N, p) \sim p^{-1/3} N^{2/3}, \quad (4)$$

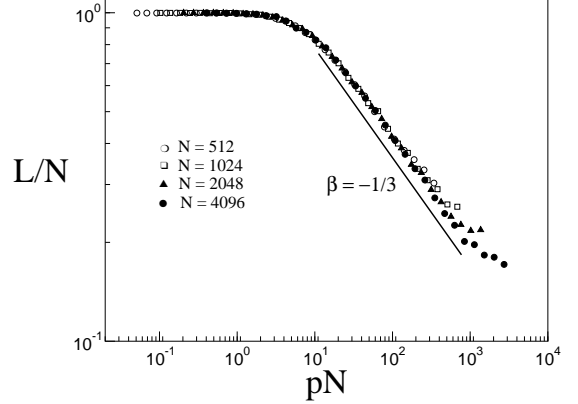


FIG. 5: Collapse of the scaling function F_2 . The line with slope $-1/3$ is a guide for the eye. The linear fitting on the scaling region $Np > 1$ gives $\beta = -0.332 \pm 0.003$.

for $pN > 1$, when loops appears. This result is particular for the present model, because in the classic SW model the mean distance scales as $\log N$ and in the present system it scales as a power-law with exponent $2/3$.

Why the model presents this SW effect in a power-law form? For our numerical experiments we have computed the Euclidian distance R in the two dimensional plane of the line linking the start with the end of the maze, the classical end-to-end distance for a RW. For a typical RW in two dimensions R scales with N as $R \sim N^{1/2}$. In figure 6 it is shown how pR scales with pN as a power-law with exponent $1/2$:

$$pR \sim (pN)^{1/2}. \quad (5)$$

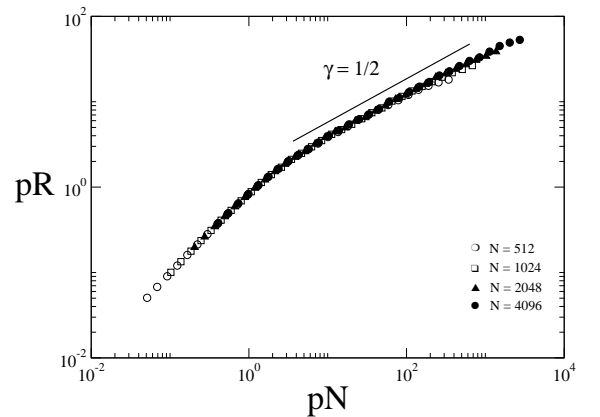


FIG. 6: Scaling relation between the end-to-end distance R and N , the number of the RW steps rescaled both by p . The line with slope $1/2$ is a guide for the eye. The scaling exponent obtained by fitting on the scaling region is $\gamma = 0.499 \pm 0.004$.

This result shows that the maze model acts as a typical

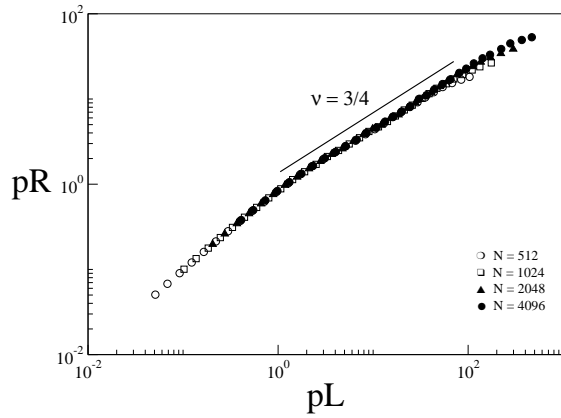


FIG. 7: Scaling relation between the end-to-end distance R and L scaled both by p . The line with slope $3/4$ is a guide for the eye. Scaling exponent obtained on the scaling region by fitting is $\nu = 0.75 \pm 0.01$.

RW model of N steps in two dimensions where N and R are scaled by p .

Self-Avoiding Random Walks (SAW) are RW where trajectories self-intersections are avoided. In our model once the RW trajectory is finished, deleting the loops

produces a SAW of length L . Its known that in a typical SAW model in two dimensions of L steps we can expected $R \sim L^{3/4}$ [15] as scaling relation. This imply in our model, that after rescaled R and L as pR and pL :

$$pR \sim (pL)^{3/4}. \quad (6)$$

In figure (7) the above scaling relationship is confirmed.

Then, after combining equations (5) and (6) we can recover equation (4). The SW effect in the two dimensional model as a power-law scaling between N and L with exponent $2/3$ for values pN , is a direct consequence of the well-known scaling relations in classical RW and SAW models.

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